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THE MAXIMUM CAPTURE PROBLEM IN A WEIGHTED NETWORK*

by

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ABSTRACT

In this paper, a conditional location problem on a weighted network is examined. Attraction functions are defined and the problem is formulated as a mixed integer program. A method which simultaneously locates an additional facility on the given network and optimizes its weight is then described. A numerical example is provided.

Key words : Voronoi diagrams; networks; location; weights.

RÉSUMÉ

Dans cet article, on décrit un problème conditionnel de localisation d'installations dans un réseau pondéré. On définit des fonctions d'attraction et on formule le problème à l'aide d'un modèle de programmation mathématique en nombres entiers. On décrit ensuite un algorithme qui permet de déterminer simultanément la localisation et le poids optimaux d'une nouvelle installation. La méthode proposée est illustrée par un exemple numérique.

Mots-clés : diagrammes de Voronoi; réseaux; localisation; poids.

1. Introduction

The purpose of this paper is to develop a model whose solution simultaneously optimizes the location of a new facility on a given network as well as one of its features called its weight. This work is rooted in closest point models usually known as Voronoi diagrams and in gravity models. The original work on Voronoi diagrams is due to Voronoi (1908) and Thiessen (1911); for a recent survey see Eiselt and Pederzoli (1986). Voronoi diagrams have a large number of applications in fields as diverse as archaeology, physics, geography and biology (for pertinent references see Aurenhammer and Edelsbrunner (1984)) and VLSI design (Meshkat and Sakkas (1987)). A variety of related problems as well as optimal algorithms for the construction of a Voronoi diagram in the plane are given by Shamos and Hoey (1976) and recently Fortune (1987).

The other foundation of this study derives from the gravity models first introduced and discussed by Reilly (1929) and Converse (1948) and later further developed by Huff (1964, 1973). Huff's "trading areas" coincide with the Voronoi areas described earlier. To our knowledge, it was Boots (1980) who introduced the weights long known to "retail gravitationalists" into the context of Voronoi diagrams. The algorithm for the determination of weighted Voronoi diagrams was developed by Aurenhammer and Edelsbrunner (1984). Models optimizing facility location and weights in the context of one-dimensional Voronoi diagrams have been studied by Eiselt and Laporte (1988a,c). The model investigated by ReVelle (1986) is simpler in that it uses equal weights for the facilities but it finds locations on a network which is a more complex space than the Eiselt and Laporte linear market.

In this study, we will generalize ReVelle's findings in two directions. Firstly, we allow differential weights for the facilities and

secondly, we will leave a parameter of the cost function variable so as to facilitate sensitivity analysis of the optimal solution. The proposed methodology is illustrated on a numerical example.

2. The Model

Consider a graph $G = (N,A,b,d)$ with sets of nodes $N = \{n_1, \ldots, n_s\}$ and arcs $A = \{a_{ij}: n_i, n_j \in N\}$. The mapping $b:N \to \mathbb{R}_+^s$ is the demand function where b_i specifies the demand (in dollars) for a homogeneous product at node n_i and the mapping $d:A \to \mathbb{R}_+^t$ is the distance function with d_{ij} being the length of a shortest path from n_i to n_j. If $i=j$, d_{ij} must be interpreted as an arbitrarily small positive constant δ. Let now $w_j > 0$ denote the weight of a facility located at n_j. This weight measures parameters as diverse as the size of the facility, its relative price advantage, the courteousness of its staff, etc. In general, a facility with a large weight is more popular among consumers than one with a small weight. If a node n_j is not occupied by a facility, then $w_j = 0$. In our model the given graph G may be directed, undirected, or mixed.

It is often assumed that customers patronize the facility which is closest to them. This assumption is justified only as long as the different facilities are comparable in their prices, convenience and other features. If differences between facilities become non-negligible, a parameter measuring these differences, here called weight, has to be introduced. Moreover, a function has to be designed measuring the trade-off between the weight of a facility and the facility-customer distance, as perceived by the customer. Here it is assumed that a customer at n_i is attracted to a facility located at n_j according to the attraction function $\phi_{ij} = w_j/d_{ij}$. Similar functions have been used in the aforementioned papers by Boots (1980) and Aurenhammer and Edelsbrunner (1984). The function is essentially a simplified version of the gravity-type models which work with squared

distances that appear in a large variety of physical contexts. A common feature of all attraction functions is that a larger weight increases the attraction of a customer to a facility whereas a smaller distance between them decreases it.

Now define y_{ij} as the proportion of the demand at node n_i which is to be satisfied by the facility located at n_j. First assume that customers at node i are evenly split between the facilities they are most attracted to and let F_i be the set of these facilities. Then

$$F_i = \{n_j \epsilon N : \phi_{ij} = \max_{k \epsilon N} \{\phi_{ik}\}\} \quad (i \epsilon N) \tag{1}$$

and

$$y_{ij} = \begin{cases} |F_i|^{-1} & (i \epsilon N, n_j \epsilon F_i) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Such a behavioral assumption does however completely ignore all attractions which are not maximal. Thus it appears more reasonable to assume that the ratios of attractions of two facilities at some demand point approximate the ratio of purchases made at the two respective facilities. Generalizing this assumption to all given facilities, we can define

$$y_{ij} = \phi_{ij} / \sum_{k \epsilon N} \phi_{ik} \quad (i,j \epsilon N) \tag{3}$$

as the proportion of the demand at node n_i which is to be satisfied by the facility located at n_j. Note that this attraction function is a special case of that used by Huff (1973). For both definitions of y_{ij}, the total demand satisfied by a facility at node n_j is $\sum_{i \epsilon N} b_i y_{ij}$.

The location model studied here is – just like that investigated by ReVelle – of the conditional type: given that a set X of p facilities in the network are at fixed locations, where should r new facilities be located ? The solution to this problem is referred to as an (r/X_p)

medianoid by Hakimi (1983). In this paper, we restrict our attention to the location of <u>one</u> additional facility (i.e. r=1), but we also seek to determine simultaneously the optimal weight of that new facility.

The problem can be formulated as a mathematical program as follows. As above, N denotes the set of all nodes. Let $M_1 \subseteq N$ be the non-empty set of existing facilities and $M_2 \subseteq N$ be the set of potential facility sites. We allow for the case where a new facility may be located on top of an existing one, i.e. M_1 and M_2 may intersect. The weight w_j of an existing facility must be interpreted as a known constant while the weight w_j of a potential facility located at node n_j is a decision variable. The cost of establishing a facility of weight w_j at site n_j is equal to cw_j where $c > 0$. Also define

$$\psi_i = \max_{j \in M_1} \{\phi_{ij}\} \qquad (i \in N)$$

$$E_i = \{j \in M_1 : \phi_{ij} = \psi_i\} \qquad (i \in N)$$

$$x_j = \begin{cases} 1 & \text{if a new facility is sited at node } n_j \\ 0 & \text{otherwise} \end{cases} \qquad (j \in M_2)$$

y_{ij}: the proportion of the demand at node n_i satisfied by a new facility located at n_j $\quad (i \in N,\ j \in M_2)$.

The formulation is then

$$(P) \quad \text{Maximize} \quad z = \sum_{j \in M_2} \left(\sum_{i \in N} b_i y_{ij} - cw_j \right) \qquad (4)$$

subject to

$$\sum_{j \in M_2} x_j = 1 \qquad (5)$$

$$y_{ij} = \begin{cases} 0 & \text{if } w_j x_j / d_{ij} < \psi_i \\ (|E_i| + 1)^{-1} & \text{if } w_j x_j / d_{ij} = \psi_i \\ 1 & \text{if } w_j x_j / d_{ij} > \psi_i \end{cases} \qquad (6)$$

$$(i \in N,\ j \in M_2)$$

or

$$y_{ij} = \frac{w_j x_j / d_{ij}}{\sum_{k \in M_1 \cup M_2} w_k x_k / d_{ik}} \qquad (i \in N,\ j \in M_2) \qquad (6')$$

$$x_j \in \{0,1\} \qquad (j \in M_2) \qquad (7)$$

$$y_{ij} \geq 0 \qquad (i \in N, j \in M_2) \qquad (8)$$

$$w_j \geq 0 \qquad (j \in M_2)\ . \qquad (9)$$

In this formulation, constraints (5) stipulate that only one new facility is to be located. Constraints (6) and (6') correspond to the cases where y_{ij} is defined by (2) and by (3) respectively. Note that since the coefficient of w_j in the objective function is negative, this variable will never take a positive value when $x_j = 0$ since it always multiplied by x_j in (6) and in (6').

First consider the model defined with constraints (6). If r new facilities are to be located instead of only one, it is then necessary to replace the right-hand side of (5) by r and to update the definition of y_{ij}. It can be shown that the model is then equivalent to that of Eiselt and Laporte (1988b) and to that of ReVelle (1986) when all w_j's are equal. In the case where r=1, Eiselt and Laporte (1988b) propose a simple solution methodology for (P). Unfortunately, the solution possesses an undesirable property: for a given location of the new facility (at node n_j, say),

the profit z is a sawtooth function of w_j. While the model may adequately describe reality between the breakpoints, it is clearly unsatisfactory in their close vicinity. This drawback can however be avoided if y_{ij} is defined by (3) instead of (2) and constraints (6') are used. This way of modeling consumer behaviour also appears more realistic and will henceforth be used.

3. Solution methodology

The model defined by using constraints (6') in (P) is a non-linear mixed integer program. It can however be solved by means of a relatively simple procedure which considers in turn all candidate locations $n_j \in M_2$. Define z_j as the profit obtained by locating the new facility at n_j (i.e. the value of z if $x_j = 1$) and substitute (6') in (4):

$$z_j = \sum_{i \in N} b_i \frac{w_j}{w_j + d_{ij} \sum_{k \neq j} w_k / d_{ik}} - c w_j \ . \tag{10}$$

Setting the derivative $\frac{dz_j}{dw_j} = 0$ yields the optimality condition

$$\sum_{i \in N} b_i \frac{d_{ij} \sum_{k \neq j} w_k / d_{ik}}{(w_j + d_{ij} \sum_{k \neq j} w_k / d_{ik})^2} = c \ . \tag{11}$$

Relation (11) is a polynomial equation in w_j of order up to n^2. The second derivative of z_j is negative for $w_j > 0$ and hence, over that domain, the left-hand side of (11) attains a maximum at $w_j = 0$. It follows that a solution of (11) exists and is unique, provided

$$c < \sum_{i \in N} \frac{b_i}{d_{ij} \sum_{k \neq j} w_k / d_{ik}} \ . \tag{12}$$

If c violates condition (12), then the optimal weight in $\bar{w}_j = 0$, resulting in a zero profit. Let again (12) be satisfied. Inserting (11) in (10) yields the maximal profit $\bar{z}_j$ and its corresponding weight $\bar{w}_j$:

$$\bar{z}_j = \sum_{i \in N} b_i \frac{\bar{w}_j^2}{(\bar{w}_j + d_{ij} \sum_{k \neq j} \bar{w}_k / d_{ik})^2} \quad . \tag{13}$$

In order to obtain a closed form expression, $\bar{z}_j$ has been written as a function of $\bar{w}_j$ even though both $\bar{z}_j$ and $\bar{w}_j$ are functions of c, as a given value of c implies an optimal weight $\bar{w}_j$ as well as a profit at optimum $\bar{z}_j$.

This procedure is repeated for all potential locations $n_j \in M_2$ and the optimal profits $\bar{z}_j$ are plotted against c on the same graph. The optimal location and weight are provided by the upper envelope of the profit curves.

To illustrate, consider the graph G = (N,A,b,d') depicted in Figure 1. The numbers next to the nodes are the demands and the numbers on the edges represent the lengths of the direct connections d'_{ij} $(=d'_{ji})$ from which the lengths d_{ij} of the shortest paths are calculated. Currently, one facility with weight one exists at n_4 and a facility with weight two exists at $n_$.

The profit function for a tentative location at n_2, say, is

$$z_2 = 40 \frac{7w_2}{7w_2 + 24} + 50 + 30 \frac{2w_2}{2w_2 + 11} + 0 + 10 \frac{12w_2}{12w_2 + 49} + 0 - cw_2$$

which yields the optimality condition

$$\frac{6\,720}{(7w_2 + 24)^2} + \frac{660}{(2w_2 + 11)^2} + \frac{5\,880}{(12w_2 + 49)^2} = c$$

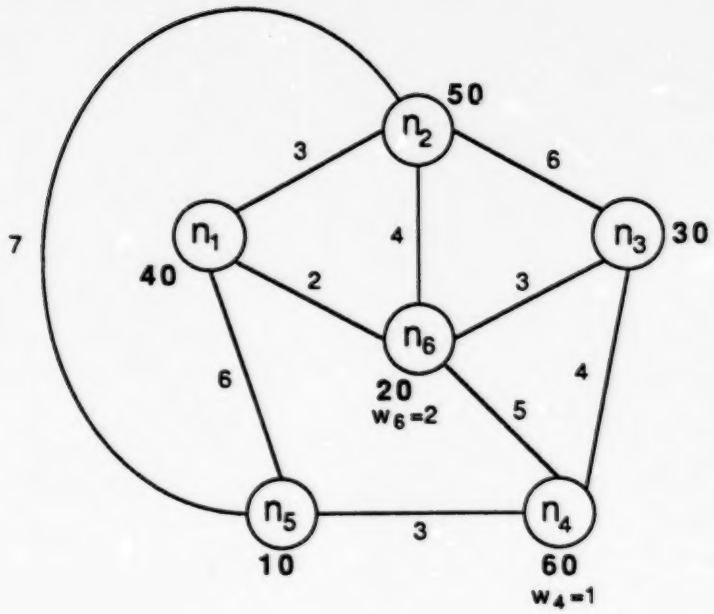

Figure 1 : A six node graph with two facilities

and the optimal profit

$$\bar{z}_2 = \frac{1\,960\,\bar{w}_2^{\,2}}{(7\bar{w}_2 + 24)^2} + 50 + \frac{120\,\bar{w}_2^{\,2}}{(2\bar{w}_2 + 11)^2} + \frac{1\,440\,\bar{w}_2^{\,2}}{(12\bar{w}_2 + 49)^2}$$

This procedure is repeated for all admissible values of j. The profit curves are displayed in Figure 2 and the corresponding decision rule is given by the upper envelope of these curves:

if $c \in [0;13.64]$: locate at n_4,

if $c \in [13.64;14.14]$: locate at n_1, and

if $c \in [14.14;\infty[$: locate at n_2.

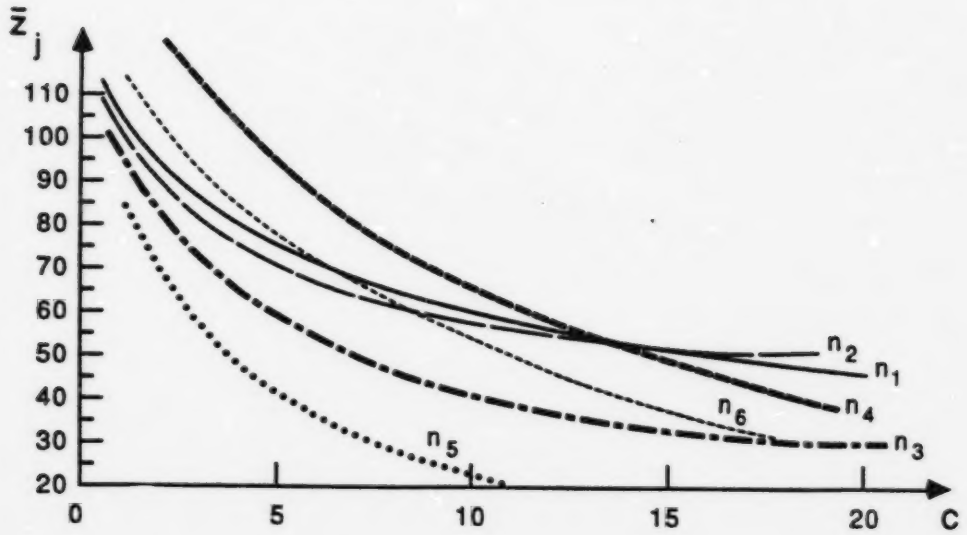

Figure 2 : Optimal profit as a function of facility cost.

4. Conclusion

In this paper, a conditional location model on a network is discussed. An attraction function is defined in terms of consumer-facility distance and facility weight. The optimal location and weight of a new facility are then determined, using a simple procedure.

The model introduced in this paper can be extended in various ways. One possibility is a locational and weight game not unlike that described by Okabe and Suzuki (1987) for equally weighted facilities in the plane. Each facility optimizes its weight and location based on the current situation as shown previously. This procedure is then repeated until either an equilibrium is reached or it becomes clear that no equilibrium exists.

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